

# Generalized Geometry and M theory.

David S. Berman  
Department of Physics,  
Queen Mary  
Mile End Road  
London E4 9NS,  
England

Malcolm J. Perry  
Department of Applied Mathematics and Theoretical Physics,  
Centre for Mathematical Sciences,  
University of Cambridge,  
Wilberforce Road,  
Cambridge CB3 0WA,  
England

and

Perimeter Institute for Theoretical Physics,  
31 Caroline St N.,  
Waterloo,  
Ontario N2L 2Y5,  
Canada.

April 28, 2011

## Abstract

We reformulate the Hamiltonian form of bosonic eleven dimensional supergravity in terms of an object that unifies the three-form and the metric. For the case of four spatial dimensions, the duality group is manifest and the metric and C-field are on an equal footing even though no dimensional reduction is required for our results to hold. One may also describe our results using the generalized geometry that emerges from membrane duality. The relationship between the twisted Courant algebra and the gauge symmetries of eleven dimensional supergravity are described in detail.

# 1 Introduction

It has been a long standing puzzle that the spacetime metric and bosonic matter fields in M theory (or string theory) appear to be on different footings. The metric provides us with a notion of spacetime. The matter fields are things that propagate in spacetime. In a theory that ought to unify all fields, it seems strange that they are thought of as so distinct. It is the case that the Kaluza-Klein picture goes some way towards resolving this issue, however in M-theory one is still left with a three form potential describing some bosonic matter whilst the metric describes the eleven dimensional spacetime.

In 1980, Julia discovered that 11-dimensional supergravity when compactified on tori of various dimensions, exhibited a host of symmetries [1], [2], which were enumerated by Morel and Thierry-Mieg [3], and by Eugene Cremmer in the special case of dimension five [4]. They are also related to the symmetries of the vacuum Einstein equations discovered by Gibbons and Hawking, [5]. They are now usually rather plainly referred to as M theory dualities and it is well known that they are controlled by some Lie group in low dimensions. In dimensions greater than 8 it seems that some more complicated object is involved. In 1986, de Wit and Nicolai found some tantalizing clues that these duality symmetries had content beyond what one expects from a theory compactified on a torus. They speculated that the duality groups somehow controlled the theory in a fundamental way, [6]. The challenge of finding the true meaning of these groups in M-theory has subsequently fascinated many authors resulting in copious publications [7]. A particular mention should be made of [8] which has similar goals to those described in section 6 of this paper.<sup>1</sup>

The present work is devoted to a reformulation of eleven dimensional supergravity (or at least its bosonic sector) to exhibit a duality symmetry without the use of any compactification. The duality symmetry is an organizing principle and places both the metric and three form in representations of the duality group itself. The duality group also controls the dynamics of the theory in a fundamental way. In some ways, the final version of the theory is rather like a  $\sigma$ -model for some symmetric space related to the duality group. However there are fundamental differences which are described towards the end of this paper. On the route to our final result we make use of the generalized geometry recently described by Hitchin [9] and by Gualtieri [10] and discussed in an M-theory context by Hull [11] and by Pacheco and Waldram [12]. We describe the dynamics of the theory using this generalized geometry. We also discuss how the generalized geometry arises from the world volume description of the M theory two-brane and from the symmetry algebra of M-theory itself.

We work with the specific case of the  $SL(5)$  duality group. This is the group that would arise in a  $T^4$  spatial reduction of M-theory. One begins in eleven dimensions with a local  $SO(10, 1)$  group and a global  $GL(11)$  associated with the spacetime metric. We then wish to

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<sup>1</sup>We apologize to those we have not cited, it is our ignorance. If you feel that some work should be cited, please let us know and we will gladly cite relevant works in the published version of this paper.

consider the subgroup of these spacetime groups which is expected to be a local  $SO(4)$  and a global  $SL(4)$  corresponding to the four dimensions of space. What happens however seems like a miracle, as there is a symmetry enhancement and the  $SL(4)$  is enlarged to an  $SL(5)$ , the local  $SO(4)$  becomes  $SO(5)$  and these groups act on both the metric and the three-form in a unified way. It is this  $SL(5)$  subgroup that we will study in detail. Let us emphasize that in this construction, no dimensional reduction has been imposed and all the fields can depend on the coordinates associated with these four spatial directions.

We suspect that one can extend our treatment to all of the exceptional duality groups that have been found via dimensional reduction. The first higher dimensional groups one encounters however will require the use of fivebranes and a more complicated generalized metric structure involving  $C_6$ , the six form potential corresponding to dual seven form field strength of M theory. If one progresses to dimension seven, then even more complicated objects must be involved. We expect to address this in future work.

We use the Hamiltonian form of the theory, following closely the work of Dirac [13]; Arnowitt, Deser and Misner [14]; and deWitt [15]. The reason we choose to work with the Hamiltonian form is because we wish to keep time separate. Were it the case that one of the directions that are acted on by the duality transformations were timelike, so that signature of the metric was not Euclidean, then the duality would be of a more exotic type and the action of the symmetry on the C-field would imply a complexification of the fluxes [16]. To avoid such difficulties, we demand that time is treated separately this is what the canonical Hamiltonian form of gravity described by Dirac [13]; Arnowitt, Deser and Misner [14]; and deWitt [15] accomplishes.

Our aim is to construct the Hamiltonian form of M theory using the metric and three form data restructured in a way that makes the  $SL(5)$  group manifest and in some sense geometric. By geometric we mean the notion of generalized geometry introduced by Hitchin [9] and others [10]. In generalized stringy geometry, the tangent space is complemented by a copy of the cotangent space at each point in space. By so doing, the  $O(d, d)$  T-duality group becomes natural, [17]. The appearance of the cotangent space is associated with the geometry of the space as seen by the string winding modes. The Kalb-Ramond field is the *twist* that mixes the two halves of big space. One half of the space is the usual geometry and the other is its T-dual. Of course one also must define a *section* on the doubled space to relate it to a particular duality frame in order to retrieve some kind of conventional spacetime picture. For the  $O(d, d)$  group of T-duality, this process is like picking a polarization on the space. The string world sheet view of this is described in [17] and the quantum consistency is discussed in [21, 22].

The generalized geometry for M-theory is a bit different. We supplement the tangent space with the space of two-forms corresponding the space seen by wrapped membranes. The three-form field  $C_{abc}$  is then the *twist* that mixes the spaces. The space is no longer simply doubled since the number of winding modes is different to the number of space dimensions for membranes. For the specific case at hand, where the number of space dimensions is 4, the

number of possible windings is 6. Thus the enlarged space that includes the usual space and its dual is 10 dimensional. Some work relating generalized geometry and branes has appeared in [23].

We show how this generalized geometric structure arises directly from considering the membrane world volume and its dual formulation pioneered by Duff and Lu [20].

Finally, armed with the geometric package from the membrane, we will construct a dynamical theory out of the generalized metric that is equivalent to the canonical Hamiltonian form of eleven dimensional supergravity. In doing so we will not assume that the theory is independent of any spacetime directions. That is, we do not carry out a dimensional reduction and we do not assume the existence of any Killing vectors.

The only assumption that we do make to show equivalence to the usual theory is to assume that the fields are independent of the dual *wrapped* membrane directions. This assumption is a kind of *section* condition describing how one takes a section through the extended space that results in our usual spacetime view.

We expect that this is not the only possible section condition. In the doubled formalism of the string [17] there are many possible ways of taking a section through the doubled space and these are of course related by T-duality transformations. The absence of a globally defined section is also possible and it is this possibility that gives rise to so called nongeometric backgrounds such as T-folds where local geometric patches are related by transition function given by T-duality transformations. We will not deal these with issues here but note that the possibility of a non-globally defined section in our enlarged space should lead to a geometric construction of *nongeometric* U-folds. Recent work using generalized geometry and U-folds has appeared in [25]. We expect to return to this question in a future paper.

We will also ignore global considerations throughout the analysis. Such global questions will be important in topologically nontrivial situations and again we leave this for further work. A related issue is that the groups described here will be real valued. Whilst this makes sense as a solution generating symmetry, we expect that only some discrete arithmetic subgroup of the duality group is expected to be an M theory symmetry, [26], [27].

The structure of the paper is as follows. After carrying out the canonical treatment of the bosonic sector of eleven dimensional supergravity we determine the generalized metric that arises in M-theory from examining the world volume dualisation of the membrane. Finally, the dynamics of this generalized metric are described and matched to the canonical form of the usual supergravity under the condition that the metric is independent of dual membrane winding mode directions. Along the way we will observe the relationship between the Courant bracket algebra, the gauge symmetries in eleven dimensional supergravity and the generalized geometry introduced by considering membrane duality transformations.

This paper covers a large area of research and as such there are many relevant and interesting works worthy of mention amongst these are [28].

## 2 The Canonical Treatment of General Relativity

The starting point is a canonical treatment of the bosonic part of M-theory. The gravitational part of the action is just the usual Einstein-Hilbert term, an integral of the Lagrangian density  $L$  over the spacetime  $\mathcal{M}$ . We follow the well worn path explored by Dirac [13], Arnowitt, Deser, Misner [14] and deWitt [15]. In  $d$  spacetime dimensions with metric tensor  $g_{ab}$ , we take this to be

$$I_{grav} = \int_{\mathcal{M}} d^d x \ L, \quad \text{with} \quad L = \sqrt{g} R \quad (1)$$

where  $g = -\det g_{ab}$  and  $R$  is the Ricci scalar (of the torsion-free metric connection) of  $g_{ab}$ . Next, we make the mild topological restriction that the spacetime can be foliated by a family of surfaces of constant time,  $\Sigma(t)$ , where  $t$  is a time co-ordinate. Let  $x^i$  be the  $\bar{d} = d - 1$  spatial co-ordinates. Now we can decompose the spacetime metric  $g_{ab}$  into a purely spatial metric of positive definite signature  $\gamma_{ij}$ , a lapse function  $\alpha$  and a shift vector  $\beta^i$  by writing

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \beta_i \beta^i & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}. \quad (2)$$

The inverse spacetime metric then becomes

$$g^{ab} = \begin{pmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i \beta^j / \alpha^2 \end{pmatrix}. \quad (3)$$

Using the above decomposition of the metric, the volume measure on spacetime  $d^d x \sqrt{g}$  is then given by  $d^{\bar{d}} x dt \alpha \sqrt{\gamma}$  where now  $\gamma = \det \gamma_{ij}$ . The second fundamental form on the surfaces  $\Sigma(t)$  is

$$K_{ij} = \frac{1}{2\alpha} (D_i \beta_j + D_j \beta_i - \dot{\gamma}_{ij}). \quad (4)$$

where now  $D$  is the covariant derivative operator formed from  $\gamma_{ij}$ , and dot denotes the partial derivative with respect to  $t$ .

Given this decomposition into space and time, we can rewrite the gravitational action as

$$I_{grav} = \int_{\mathcal{M}} d^{\bar{d}} x dt \alpha \gamma^{\frac{1}{2}} (R(\gamma) + K_{ij} K^{ij} - K^2) \quad (5)$$

$R(\gamma)$  is the Ricci scalar formed from the spatial metric  $\gamma$ , and  $K$ , the trace of the second fundamental form, is  $K = K_{ij} \gamma^{ij}$ . In the derivation of the above expression for  $I_{grav}$ , boundary terms have been discarded.<sup>2</sup>

We now proceed to find the Hamiltonian for this theory. Firstly we must identify the canonical momenta conjugate to the fundamental variables of the theory. These we denote by

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<sup>2</sup>Throughout this work we have neglected surface terms. These terms are potentially important in topologically nontrivial situations and we leave such analysis for the future.

$\pi_\alpha$ ,  $\pi_\beta^i$  and  $\pi_\gamma^{ij}$  which are conjugate to  $\alpha$ ,  $\beta_i$  and  $\gamma_{ij}$  respectively. One immediately finds two primary constraints,  $\pi_\alpha \approx 0$  and  $\pi_\beta^i \approx 0$ , the momenta conjugate to the lapse and the shift vector both vanish weakly. The momentum conjugate to the spatial metric is given by

$$\pi_\gamma^{ij} = \gamma^{1/2} \left( -K^{ij} + \gamma^{ij} K \right). \quad (6)$$

It should be noted that all momenta are densities of weight one with respect to the spatial metric  $\gamma_{ij}$ . The Hamiltonian is then

$$H = \int_{\Sigma(t)} d\bar{x} \left( \pi_\alpha \dot{\alpha} + \pi_\beta^i \dot{\beta}_i + \pi_\gamma^{ij} \dot{\gamma}_{ij} - L \right) \quad (7)$$

where  $L$  is the Lagrangian density. Requiring the primary constraints to be preserved under time evolution results in two secondary constraints, the diffeomorphism constraint

$$\chi^i = -2D_j \pi^{ij} \quad (8)$$

and the Hamiltonian constraint

$$\mathcal{H} = \gamma^{-1/2} \left( \pi^{ij} \pi_{ij} - \frac{1}{d-1} \pi^2 - \gamma R \right), \quad (9)$$

both of which are also required to vanish weakly. The Hamiltonian can now be written as

$$H = \int d\bar{x} \left( \pi_\alpha \dot{\alpha} + \pi_\beta^i \dot{\beta}_i + \alpha \mathcal{H} + \beta_i \chi^i \right). \quad (10)$$

It should be noted that the complete Hamiltonian weakly vanishes. Each of these constraints is first class, and so corresponds to a gauge degree of freedom. Each constraint must be supplemented by a gauge condition so that the Poisson bracket of all of the constraints and gauge conditions has non-vanishing determinant. A consequence of the gauge fixing is that Hamiltonian evolution of the system will preserve not only the constraints but the gauge conditions. Thus imposing them at one instant will ensure that they are imposed for all time.

It is worth our while counting up the physical degrees of freedom to check that they come out correctly. The lapse, shift and spatial metric together with their conjugate momenta give  $2 + 2\bar{d} + \bar{d}(\bar{d} + 1)$  degrees of freedom per point in space. The constraints give  $1 + \bar{d} + 1 + \bar{d}$  for  $\pi_\alpha$ ,  $\pi_\beta^i$ ,  $\mathcal{H}$  and  $\chi^i$  respectively and the same for each gauge condition. This leads to the physical phase space having dimension  $(\bar{d} + 1)(\bar{d} - 2)$  at each point in space, as one expects for the Einstein gravity.

Let us now study what this Hamiltonian means for the gravitational field. Firstly we are free to make a gauge choice for the lapse and shift. We can choose the synchronous gauge  $\alpha = 1$  and  $\beta^i = 0$  for definiteness and simplicity, but it does not really matter what choice we

make. There are still  $\bar{d} + 1$  gauge conditions that need to be chosen to restrict  $(\gamma_{ij}, \pi^{ij})$ . Now look at the Poisson bracket algebra of the diffeomorphism and Hamiltonian constraints.

$$\{\mathcal{H}(x), \mathcal{H}(x')\} = (\chi^i(x) + \chi^i(x'))D_i\delta(x, x'), \quad (11)$$

$$\{\chi_i(x), \mathcal{H}(x')\} = \mathcal{H}(x)D_i\delta(x, x') \quad (12)$$

and

$$\{\chi_i(x), \chi_j(x')\} = \chi_i(x')\partial_j\delta(x, x') + \chi_j(x)\partial_i\delta(x, x') \quad (13)$$

The algebra shows that if one imposes the Hamiltonian constraint everywhere, the diffeomorphism constraint is automatically satisfied. That comes about because if the Hamiltonian constraint is satisfied, then the Poisson bracket of the Hamiltonian with itself must also vanish, and hence the diffeomorphism constraint is also satisfied by virtue of (11), as was first demonstrated by Moncrief and Teitelboim, [29]. For this reason, we will focus only on the Hamiltonian constraint. The Hamiltonian constraint contains two pieces, a kinetic piece that is quadratic in the momenta and a potential term that is proportional to the curvature.

The kinetic piece can be written

$$T = \mathfrak{G}_{ijkl}\pi^{ij}\pi^{kl} \quad (14)$$

with

$$\mathfrak{G}_{ijkl} = \frac{1}{2}\gamma^{-1/2} \left( \gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \frac{2}{\bar{d}-1}\gamma_{ij}\gamma_{kl} \right), \quad (15)$$

and a potential piece

$$V = \gamma^{1/2}R. \quad (16)$$

If  $T$  were the only term in the Hamiltonian, then each point in space would behave independently. The constraint would then be what one expects for the Hamiltonian of free particle motion on a space  $M$  of dimension  $\frac{1}{2}\bar{d}(\bar{d}+1)$  at each point  $x$  in space. The momenta  $\pi^{ij}$  behave as components of a covariant vector in this space, each pair of indices  $i \leq j$  behaving as one index in  $M$ .  $\mathfrak{G}_{ijkl}$  thus is the inverse metric on the space  $M$ . The metric on  $M$ ,  $\bar{\mathfrak{G}}^{ijkl}$  is consequently determined by

$$\mathfrak{G}_{ijkl}\bar{\mathfrak{G}}^{klmn} = \frac{1}{2}(\delta_i^m\delta_j^n + \delta_i^n\delta_j^m), \quad (17)$$

to be

$$\bar{\mathfrak{G}}^{klmn} = \frac{1}{2}\gamma^{1/2} \left( \gamma^{km}\gamma^{ln} + \gamma^{kn}\gamma^{lm} - 2\gamma^{kl}\gamma^{mn} \right). \quad (18)$$

It should be noted that the inverse metric on  $M$  is not found by raising the indices of  $\mathfrak{G}$  using the spatial metric. The signature of  $\bar{\mathfrak{G}}^{ijkl}$  is  $(+\frac{1}{2}(\bar{d}+2)(\bar{d}-1), -)$  for  $\bar{d} > 1$ . One can think of the

metric on  $M$  as being a metric on the space of all symmetric tensors at each point in space. Suppose one has a symmetric tensor  $h_{ij}$  at some point in space. Then the norm of  $h_{ij}$  is

$$\|h\|^2 = \mathfrak{G}^{ijkl} h_{ij} h_{kl}. \quad (19)$$

If  $h_{ij}$  were to be proportional to the spatial metric tensor,  $h_{ij} = \phi \gamma_{ij}$  then  $\|h\|^2 = -\gamma^{1/2} \phi^2 \bar{d}(\bar{d}-1) < 0$  provided we are in spacetime dimension  $d > 2$ . It is only for  $d > 3$  that there is any meaningful idea of dynamical gravitation. Alternatively one can think of the metric components  $\gamma_{ij}$  as providing a set of co-ordinates on  $M$ . There is a single negative (timelike) direction that corresponds to conformal transformations of the spatial metric  $\gamma_{ij}$ . The fact that  $M$  does not have a positive definite metric is related to bad behavior of the Euclidean path integral where it has been found that the Euclidean Einstein action is unbounded below, [30].

We need to explore the geometry of  $M$ , so firstly define on  $M$  a co-ordinate  $\zeta = 4\sqrt{(\bar{d}-1)/\bar{d}}\gamma^{1/4}$ .  $\zeta$  is timelike. Suppose that there are a further set of co-ordinates  $\xi^A$ , ( $A = 1, \dots, \frac{1}{2}(\bar{d}+2)(\bar{d}-1)$ ) orthogonal  $\zeta$ . Then the metric on  $M$  can be written as

$$G_{AB} = \begin{pmatrix} -1 & 0 \\ 0 & \frac{\bar{d}}{16(\bar{d}-1)}\zeta^2 \bar{G}_{AB} \end{pmatrix} \quad (20)$$

where  $\bar{G}_{AB}$  is a metric on  $\bar{M}$ , the surfaces of constant  $\zeta$ . Writing the metric in this form shows us that  $M$  is a lorentzian cone over  $\bar{M}$ . The scale of  $\bar{M}$  is set by  $\zeta$ . One can calculate the curvature of  $\bar{M}$ . One finds that  $\bar{M}$  is an Einstein space of negative curvature, with

$$\bar{R}_{AB} = -\frac{\bar{d}}{2}\bar{G}_{AB}. \quad (21)$$

One also finds that the Riemann tensor is covariantly constant. Therefore  $\bar{M}$  is locally a symmetric space  $SL(\bar{d})$  acts transitively on  $\bar{M}$ , and the isotropy group is  $SO(\bar{d})$ . Thus  $\bar{M}$  can be identified with coset space  $SL(\bar{d})/SO(\bar{d})$ . One may also easily calculate the curvature of  $M$ . The Ricci scalar of  $M$  is a constant times  $\zeta^{-2}$ . Thus  $M$  is singular at  $\zeta = 0$  where the scale of the embedded  $\bar{M}$  vanishes.

Now we construct the evolution equations in the synchronous gauge. The Hamiltonian is

$$H = \int d^{d-1}x \mathcal{H} = \int d^{\bar{d}}x \left( \mathfrak{G}_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{\gamma} R \right). \quad (22)$$

The evolution equations are now found by computing the appropriate Poisson brackets. Firstly, the evolution of the metric is given by

$$\dot{\gamma}_{ij} = \{\gamma_{ij}, H\} = \frac{1}{2}\gamma^{-1/2} \left( \pi_{ij} - \frac{1}{\bar{d}-1}\gamma_{ij}\pi \right). \quad (23)$$

Recalling the definition of the momentum in terms of the second fundamental form, we easily see that this equation just amounts to a restatement of the definition of momentum. The evolution equation for the momentum is a little more complicated.

$$\dot{\pi}^{ij} = \{\pi^{ij}, H\} = \frac{1}{2}\gamma^{ij}\mathfrak{G}_{klmn}\pi^{kl}\pi^{mn} - 2\gamma^{-1/2}(\pi^{ik}\pi^j{}_k - \frac{1}{\bar{d}-1}\pi\pi^{ij} - \gamma^{1/2}(R^{ij} - \frac{1}{2}\gamma^{ij}R)). \quad (24)$$

The last term involving the Einstein tensor comes from the potential energy term and is only the only effect the potential energy term has on the evolution. We can rewrite these two equations as an evolution equation for the metric.

$$\ddot{\gamma}_{ij} - \dot{\gamma}_{ik}\dot{\gamma}_{jl}\gamma^{kl} + \frac{1}{4(\bar{d}-1)}(\gamma^{kn}\gamma^{lm} - \gamma^{kl}\gamma^{mn})\dot{\gamma}_{kl}\dot{\gamma}_{mn}\gamma_{ij} = -\frac{1}{2}\dot{\gamma}_{ij}\gamma^{mn}\dot{\gamma}_{mn} - \frac{1}{2}(R_{ij} - \frac{1}{2(\bar{d}-1)}R\gamma_{ij}). \quad (25)$$

(25) is rather like the geodesic equation. The last term on the right hand side is a kind of force coming from the curvature. If it were absent, (25) would be the geodesic equation for  $\gamma_{ij}$  in the space  $M$  at each point in physical space. We note that  $t$  is not an affine parameter which is the reason for the first term on the right hand side of the equation. One may easily parameterize the curve in  $M$  so that it is affinely parameterized. Let  $s$  be an affine parameter along the curve. It is related to  $t$  by

$$\frac{\ddot{s}}{\dot{s}} = -\frac{1}{2}\gamma^{ij}\dot{\gamma}_{ij} = -\frac{1}{2}\det\gamma \quad (26)$$

One can now explicitly solve this equation to discover that

$$\dot{s} = \frac{\text{constant}}{\zeta^2} \quad (27)$$

where  $\zeta$  is the timelike coordinate on  $M$ . Since  $\zeta = \lambda\gamma^{1/4}$  we see that time is essentially specified by the volume of space, a fact familiar to cosmologists. Each point in space would have a metric that evolves independently of every other point. This is the explanation of the billiards picture. The curvature term however has the effect of coupling different points in space to each other.

It should be noted that if one compactified space on a torus, then of course there is no spatial curvature. Additionally, each point in space is then equivalent to every other point and so the evolution of the spacetime is given by a single equation. The evolution equation is then just that of a particle moving in the symmetric space  $\bar{M}$ . The motion of such a particle is integrable. Different solutions of the geodesic equation are then related to each other by  $SL(\bar{d})$  transformations which demonstrates the origin and nature of the duality group for general relativity. The duality group for general relativity was first discovered in a somewhat different context by Buchdahl [31], and by Ehlers [32] and discussed by Geroch [33].

Of course, if curvature is introduced, it seems as if our simple picture will be spoilt by the fact that different points in space obey different equations as the curvature will depend where in space one is.

Nevertheless, the observation that  $\bar{M}$  is the same as the general relativity duality group motivates us to look further and see if it is possible to formulate similar ideas in  $M$  theory.

One can also construct a Lagrangian formulation for the evolution equation. It is found by evaluating the canonical momentum in terms of the metric and switching the sign of the potential term. Thus the Lagrangian is the relatively simple looking expression

$$I = \int d^d x dt (\bar{\mathfrak{G}}^{ijkl} \dot{\gamma}_{ij} \dot{\gamma}_{kl} + \sqrt{\gamma} R) \quad (28)$$

Suppose one makes a global  $SL(\bar{d})$  transformation on the metric, so that

$$\gamma_{ij} \rightarrow \gamma'_{ij} = M_i{}^k M_j{}^l \gamma_{kl} \quad (29)$$

with  $M \in SL(\bar{d})$ . The action is then invariant. Since the action is not scale invariant, it is not possible to extend this symmetry to  $GL(\bar{d})$ . Part of our aim is generalize this observation to  $M$  theory.

### 3 The Canonical treatment of M-theory

In  $M$ -theory, a three form abelian gauge field is coupled to gravity and the system is then treated in an entirely similar way. The new part of the action consists of two pieces; the standard kinetic term  $I_{minimal}$  which is defined in any number of spacetime dimensions and a Chern-Simons like term  $I_{CS}$  which can only be defined in eleven spacetime dimensions.

$$I_{minimal} = - \int_{\mathcal{M}} d^d x \sqrt{g} \frac{1}{48} F_{abcd} F^{abcd} \quad (30)$$

where the field strength  $F_{abcd}$  is made from the exterior derivative of a potential  $C_{abc}$  where

$$F_{abcd} = 4\partial_{[a} C_{bcd]} \quad (31)$$

$I_{minimal}$  has an abelian gauge invariance under which

$$C_{abc} \rightarrow C_{abc} + 3\partial_{[a} \Lambda_{bc]} \quad (32)$$

The Chern-Simons term is

$$I_{CS} = \lambda \int_{\mathcal{M}} d^d x \eta^{abcdefghijklm} C_{abc} F_{def} F_{hijk} \quad (33)$$

where  $\eta$  is the eleven-dimensional alternating tensor density. For M-theory,  $\lambda$  is determined by supersymmetry to be  $\lambda = 2^{-7}3^{-4}$ . It is possible to choose the opposite sign for this term but that is equivalent to changing the sign of  $C_{abc}$ ; we have made the arbitrary choice for  $\lambda$  to be positive. The Chern-Simons term is also gauge invariant, but only up to boundary terms that are irrelevant to the discussions presented here. In what follows, in  $d \neq 11$  spacetime dimensions we will take  $\lambda = 0$ , but if  $d = 11$ , then  $\lambda \neq 0$ .

Just as for the gravitational field, we split  $C_{abc}$  up into its purely spatial components and the remainder. Thus,

$$C_{abc} \rightarrow C_{ijk} \quad \text{and} \quad C_{0ij} = B_{ij}. \quad (34)$$

Then  $C_{ijk}$  has a field strength, analogous to a magnetic field, of

$$F_{ijkl} = 4\partial_{[i}C_{jkl]}, \quad (35)$$

and  $B_{ij}$  has a field strength, analogous to an electric field, of

$$G_{ijk} = 3\partial_{[i}B_{jk]}. \quad (36)$$

We now split up the action for the three-form theory using the space and time decomposition of both the metric and three-form fields to find

$$\begin{aligned} I = \int_{\mathcal{M}} d^{\bar{d}}x dt \alpha \gamma^{1/2} & - \frac{1}{48} \left( F_{ijkl} F^{ijkl} - \frac{4}{\alpha^2} \beta^i \beta^m F_{ijkl} F_m{}^{jkl} - \frac{4}{\alpha^2} \dot{C}_{ijk} \dot{C}_{lmn} \gamma^{il} \gamma^{jm} \gamma^{kn} \right. \\ & + \frac{8}{\alpha^2} \dot{C}_{ijk} G_{ijk} - \frac{4}{\alpha^2} G_{ijk} G^{ijk} + \frac{8}{\alpha^2} \beta^l \dot{C}_{ijk} F_l{}^{ijk} - \frac{8}{\alpha^2} \beta^l G_{ijk} F_l{}^{ijk} \Big) \\ & + \lambda \eta^{ijklmnnpqrs} \left( 3B_{ij} F_{klmn} F_{pqrs} - 8C_{ijk} \dot{C}_{lmn} F_{pqrs} + 8C_{ijk} G_{lmn} F_{pqrs} \right) \end{aligned} \quad (37)$$

The next step is to find the canonical momenta  $\pi_B^{ij}$  conjugate to  $B_{ij}$  and  $\pi_C^{ijk}$  conjugate to  $C_{ijk}$ . We find a primary constraint that  $\pi_B^{ij} \approx 0$  as the Lagrangian is independent of  $\dot{B}_{ij}$ . The momentum conjugate to  $C_{ijk}$  is

$$\pi_C^{ijk} = \frac{\gamma^{1/2}}{6\alpha} \left( \dot{C}_{lmn} \gamma^{il} \gamma^{jm} \gamma^{kn} - G^{ijk} - \beta^l F_l{}^{ijk} \right) + 8\lambda \eta^{ijklmnnpqrd} C_{lmn} F_{pqrs} \quad (38)$$

In terms of the canonical variables, the Hamiltonian for the three-form field is

$$\begin{aligned} H = \int_{\Sigma} d^{\bar{d}}x \pi_B^{ij} \dot{B}_{ij} + 3\alpha \gamma^{-1/2} \pi_{ijk} \pi^{ijk} + \pi^{ijk} (G_{ijk} + \beta^l F_{lij}) + \frac{\alpha \gamma^{1/2}}{48} F_{ijkl} F^{ijkl} \\ + \frac{\lambda}{54} \alpha \gamma^{-1/2} \eta_{ijklmnnpqrs} \eta^{ijk} {}_{tuvwxyz} C^{lmn} F^{pqrs} C^{tuv} F^{wxyz} \\ + \lambda \eta^{ijklmnnpqrs} (-3B_{ij} F_{klmn} F_{pqrs} + 8C_{ijk} \beta^t F_{tlmn} F_{pqrs} + 48C_{ijk} \pi_{lmn} F_{pqrs}). \end{aligned} \quad (39)$$

There is a new secondary constraint <sup>3</sup> resulting from the gauge invariance of the three-form potential

$$\chi^{ij} = -\sqrt{2}(\partial_k \pi^{ijk} + \lambda \eta^{ijklmnpqrs} F_{klmn} F_{pqrs}). \quad (40)$$

This new constraint we will call the gauge constraint.

The gauge constraint is reducible, meaning that not all of its components are independent. It obeys the identity

$$\partial_i \chi^{ij} = 0. \quad (41)$$

The Hamiltonian for the three-form theory is then

$$H_C = \int_{\Sigma} d^{\bar{d}}x \pi_B^{ij} \dot{B}_{ij} + \frac{\sqrt{2}}{3} B_{ij} \chi^{ij} + 3\alpha \gamma^{-1/2} \pi^{ijk} \pi_{ijk} + \frac{1}{48} \alpha \gamma^{1/2} F^{ijkl} F_{ijkl} \\ + \pi^{ijk} \beta^l F_{ijkl} + 8\lambda \eta^{ijklmnpqrs} \beta^t C_{ijk} F_{tlmn} F_{pqrs} \quad (42)$$

The Hamiltonian for the three-form can now be combined with the gravitational Hamiltonian to produce an expression for the complete bosonic part of M-theory.

$$H = \int_{\Sigma} d^{\bar{d}}x \pi_{\alpha} \dot{\alpha} + \pi_{\beta}^i \dot{B}_{ij} + \pi_B^{ij} \dot{B}_{ij} + \beta_i \bar{\chi}^i + \alpha \bar{\mathcal{H}} + \frac{3}{\sqrt{2}} B_{ij} \chi^{ij}. \quad (43)$$

The inclusion of the three-form has modified the diffeomorphism constraint,  $\chi^i \rightarrow \bar{\chi}^i$ . The modified constraint is

$$\bar{\chi}^i = -2D_j \pi^{ij} + F^{ijkl} \pi_{jkl} - 8\lambda \eta^{ijklmnpqrst} F_{jkl}^i C_{mnp} F_{qrst} \quad (44)$$

Similarly, the Hamiltonian constraint is modified  $\mathcal{H} \rightarrow \bar{\mathcal{H}}$  so that the modified constraint is

$$\bar{\mathcal{H}} = \gamma^{-1/2} \left( \pi^{ij} \pi_{ij} - \frac{1}{\bar{d}-1} \pi^2 + 3\pi^{ijk} \pi_{ijk} + \gamma(-R + \frac{1}{48} F^{ijkl} F_{ijkl}) \right). \quad (45)$$

All of the equations can be made to look nicer by making the following canonical transformation

$$\hat{\pi}^{ijk} = \pi^{ijk} - 8\lambda \eta^{ijklmnpqrs} C_{lmn} F_{pqrs} \quad (46)$$

so that constraints are now

$$\bar{\chi}^i = -2D_j \pi^{ij} + F^{ijkl} \hat{\pi}_{jkl} \quad (47)$$

$$\bar{\mathcal{H}} = \gamma^{-1/2} \left( \pi^{ij} \pi_{ij} - \frac{1}{\bar{d}-1} \pi^2 + 3\hat{\pi}^{ijk} \hat{\pi}_{ijk} + \gamma(-R + \frac{1}{48} F^{ijkl} F_{ijkl}) \right). \quad (48)$$

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<sup>3</sup>We have made our conventions consistent throughout the paper. One consequence is that various numerical factors look rather strange, in particular those associated with the normalization of the gauge constraint and in section five where the C field is discussed

and

$$\chi^{ij} = -\sqrt{2}(\partial_k \hat{\pi}^{ijk} + 3\lambda \eta^{ijklmnpqrs} F_{klmn} F_{pqrs}). \quad (49)$$

The Hamiltonian then remains the same except that one uses the new form for each of the constraints. It can now be seen that the overall structure of the theory is universal and independent of whether the Chern-Simons term is present or not. The Hamiltonian still consists of two parts, a kinetic piece  $T$  given by

$$T = \gamma^{-1/2} \left( \pi^{ij} \pi_{ij} - \frac{1}{\bar{d}-1} \pi^2 + 3\hat{\pi}^{ijk} \hat{\pi}_{ijk} \right), \quad (50)$$

and a potential piece  $V$  given by

$$\gamma^{1/2} \left( -R + \frac{1}{48} F^{ijkl} F_{ijkl} \right). \quad (51)$$

We should check that we have the correct number of physical degrees of freedom in our theory. The counting for the gravitational sector is unchanged. For  $B_{ij}$  and  $C_{ijk}$  the number of degrees of freedom are  $\frac{1}{2}\bar{d}(\bar{d}-1)$  and  $\frac{1}{6}\bar{d}(\bar{d}-1)(\bar{d}-2)$  respectively. Doubling this total results in a phase space that has dimension  $\frac{1}{3}\bar{d}(\bar{d}^2-1)$ . From the vanishing of  $\pi_B^{ij}$  and the conjugate gauge conditions, one subtracts  $\bar{d}(\bar{d}-1)$ . From the vanishing of  $\chi^{ij}$  one might at first sight think that there are a further  $\frac{1}{2}\bar{d}(\bar{d}-1)$  constraints. However, the fact that  $\chi_{ij}$  is identically divergence-free means that there are in fact fewer independent equations here. The divergence-free condition seems to subtract off  $\bar{d}$  from the number of conditions since there are  $\bar{d}$  vector fields. However our counting is still not quite right as one of these  $\bar{d}$  conditions is redundant coming as it does from a vector that is itself divergence free. So the correct number to subtract off is  $\bar{d}-1$ . There are therefore  $\frac{1}{2}(\bar{d}-1)(\bar{d}-2)$  independent equations in  $\chi_{ij}=0$ . Each of the independent constraints is still first class and so must be supplemented by a gauge condition. One therefore subtracts off a further  $(\bar{d}-1)(\bar{d}-2)$  to obtain the number of physical degrees of freedom. Thus, the physical phase space of the theory has dimension  $\frac{1}{3}(\bar{d}-1)(\bar{d}-2)(\bar{d}-3)$ . The rather complicated degree of freedom counting will lead in a quantum version of this theory to ghosts for ghosts and ghosts for ghosts for ghosts [34] in addition to just the usual ghosts.

Now we will examine in detail the Poisson bracket algebra for three forms coupled to gravity. For convenience, we will put  $\lambda=0$  in what follows. the Poisson brackets for three-forms coupled to gravity have been discussed in the literature previously by Baulieu and Henneaux, [35]. An involved calculation reveals

$$\{\bar{\mathcal{H}}(x), \bar{\mathcal{H}}(x')\} = (\bar{\chi}^i(x) + \bar{\chi}^i(x')) D_i \delta(x, x'), \quad (52)$$

$$\{\bar{\chi}_i(x), \bar{\mathcal{H}}(x')\} = \bar{\mathcal{H}}(x) D_i \delta(x, x') + \frac{\sqrt{2}}{18} \pi_{ijk} \chi^{jk} \gamma^{-1/2} \delta(x, x'), \quad (53)$$

$$\{\bar{\chi}_i(x), \bar{\chi}_j(x')\} = \bar{\chi}_i D_j \delta(x, x') + \bar{\chi}_j(x) D_i \delta(x, x') + \frac{1}{\sqrt{2}} F_{ijkl} \bar{\chi}^{kl} \delta(x, x'), \quad (54)$$

and

$$\{\bar{\mathcal{H}}(x), \chi^{ij}(x')\} = \{\bar{\chi}_i(x), \chi^{jk}(x')\} = \{\chi^{ij}(x), \chi^{kl}(x')\} = 0. \quad (55)$$

The first thing to note about the above algebra is that the argument of Moncrief and Teitelboim [29] is still valid. If  $\bar{\mathcal{H}}$  vanishes everywhere, then the Poisson bracket of  $\bar{\mathcal{H}}$  with itself vanishes everywhere which implies that the diffeomorphism constraint holds. Similarly the second Poisson bracket relation then implies that the gauge constraint holds. Next, notice that the diffeomorphism constraint together with the gauge constraint is a sub-algebra of the entire algebra. Consequently, one expects this sub-algebra to control the behavior of the fields in the theory under purely spatial diffeomorphisms and gauge transformations.

Gauge transformations of the three-form are easy to compute. One finds the gauge transformation of any field by computing its Poisson bracket with  $\int dx \xi_{ij}(x) \frac{3}{\sqrt{2}} \chi^{ij}(x)$ . Thus one finds the gauge variation of  $C_{ijk}$  to be

$$\delta C_{ijk} = \{C_{ijk}(x), \frac{3}{\sqrt{2}} \int dx' \xi_{lm}(x') \chi^{lm}(x')\} = 3\partial_{[i}\xi_{jk]} \quad (56)$$

If  $\xi$  is exact,  $\xi_{jk} = 2\partial_{[j}\lambda_{k]}$  for some  $\lambda$ , then the gauge transformation vanishes which reflects the fact that  $\chi^{ij}$  is reducible. If one performs a gauge transformation on any of the fields using an exact gauge parameter, then the variation is identically zero. Consequently, if one tries to determine the algebra of symmetries of a theory just from looking at the action of the constraints on the fields of the theory, as we have done here, there is a danger of missing such terms in the resultant symmetry algebra describing the algebra of gauge transformations.

One naively expects to find that under spatial co-ordinate transformations generated by vector fields  $X^i$ , a field  $F$  will transform by a quantity related to the Lie derivative of  $F$ , by  $\delta F = \mathbf{L}_X F$ . One can carry out this calculation explicitly for our diffeomorphism constraint. A co-ordinate transformation generated by  $X^i$  is found by taking the Poisson bracket of any field with  $\int dx X^i(x) \bar{\chi}_i(x)$ . Thus the variation of the metric is

$$\delta \gamma_{ij}(x) = \{\gamma_{ij}(x), \int dx' X^k(x') \bar{\chi}_k(x')\} = D_i X_j + D_j X_i \quad (57)$$

as one expects. The variation of  $C_{ijk}$  is calculated in the same way and one finds

$$\delta C_{jkl} = X^i F_{ijkl} \quad (58)$$

(58) is not what one would have expected as it is a combination of the Lie derivative and a gauge transformation

$$\delta C_{jkl} = (\mathbf{L}_X C)_{jkl} - 3\partial_{[j} \left( X^m C_{kl]m} \right) \quad (59)$$

The final point we wish to emphasize about the constraint algebra can be seen by looking at the Poisson bracket of two diffeomorphism constraints. It involves a term proportional to the four-form field strength times the gauge constraint. such a term is somewhat unexpected as it implies that the algebra of symmetry transformations is field dependent. There is no problem in principle with such a situation but this kind of thing is likely to lead to difficulties with the Jacobi identity.

The end-point of this section is then to note that the structure of the Hamiltonian constraint is rather similar to that for pure gravity except that it contains both the three-form and the metric. However, it does not have any obvious structure involving any coset space, even after the timelike direction in the kinetic term has been factored out. In section six, we will return to this point and show that there is indeed a hidden coset space construction, exactly paralleling the case for pure gravity.

## 4 Duality for the M2-brane

In this section we follow closely the work of Duff and Lu [20]. One of the fundamental constituents of M-theory is the *M2*-brane. Suppose that the *M2* is embedded in a spacetime with metric  $g_{ab}$  and three-form  $C_{abc}$ . The location of the *M2* in spacetime is given by  $X^a(\xi)$  where  $\xi^\mu$  are world-volume co-ordinates and the world-volume metric is  $h_{\mu\nu}$  of signature  $(- + +)$ . The bosonic part of the *M2* action is

$$I = \int d^3\xi \sqrt{-h} \left( \frac{1}{2} h^{\mu\nu} \partial_\mu X^a \partial_\nu X^b g_{ab} + \frac{1}{6} \epsilon^{\mu\nu\rho} \partial_\mu X^a \partial_\nu X^b \partial_\rho X^c C_{abc} - \frac{1}{2} \right). \quad (60)$$

where  $h = \det h_{\mu\nu}$  and  $\epsilon^{\mu\nu\rho}$  denotes the alternating tensor.  $\kappa$ -symmetry for the supersymmetric extension of this action requires the *M2* to propagate in a background spacetime that satisfies the classical equations of motion precisely as described in section two, [37].

The *M2* brane exhibits a duality symmetry that is somewhat similar to that found for the string. Suppose that the string is compactified on a  $d$ -dimensional torus, then it exhibits an  $SO(d, d)$  symmetry. This  $SO(d, d)$  symmetry has been related to doubled geometry which has given some insight into the nature of string theory. In this section, we will examine duality symmetry for the *M2*-brane. Suppose for the moment that we compactify on a  $d$ -dimensional torus, so that there are  $d$  commuting Killing vectors. The metric and three-form fields will be independent of the  $X^a$  associated with these Killing vectors. we require the above condition is in order to find the conventional duality symmetry. Suppose in addition that there are no other directions in spacetime in which the *M2*-brane is moving. Under such simplifying assumptions, the equations of motion that follow from the action (60) are

$$h_{\mu\nu} = \partial_\mu X^a \partial_\nu X^b g_{ab} \quad (61)$$

for the world-volume metric and

$$\partial_\mu \mathcal{G}_a^\mu = 0 \quad (62)$$

where

$$\mathcal{G}_a^\mu = (\sqrt{-h}g_{ab}\mathcal{F}_\mu^b + \frac{1}{\sqrt{2}}C_{abc}\tilde{\mathcal{G}}^{\mu bc}), \quad (63)$$

$$\mathcal{F}_\mu^a = \partial_\mu X^a, \quad (64)$$

and

$$\tilde{\mathcal{G}}^{\mu ab} = \frac{1}{\sqrt{2}}\sqrt{-h}\epsilon^{\mu\nu\rho}\mathcal{F}_\nu^a\mathcal{F}_\rho^b. \quad (65)$$

The last equation, (65), has vanishing divergence as a consequence of (64). Thus

$$\partial_\mu \tilde{\mathcal{G}}^{\mu ab} \equiv 0. \quad (66)$$

This equation is therefore the Bianchi identity. The physical content of the Bianchi identity can be seen by finding its solutions. At least locally, they are given by

$$\mathcal{F}_\mu^a = \partial_\mu X^a \quad (67)$$

for some  $X^a$ . In fact, this last result will hold independently of whether there are any Killing vectors or not.

One can now construct a first order action for the  $M2$ -brane. The Lagrangian density is

$$L = -\sqrt{-h}\left(\frac{1}{2}h^{\mu\nu}\mathcal{F}_\mu^a\mathcal{F}_\nu^bg_{ab} + \frac{1}{3}\epsilon^{\mu\nu\rho}\mathcal{F}_\mu^a\mathcal{F}_\nu^b\mathcal{F}_\rho^cC_{abc} - \partial_\mu X^a(h^{\mu\nu}\mathcal{F}_\nu^b g_{ab} + \frac{1}{2}\epsilon^{\mu\nu\rho}\mathcal{F}_\nu^b\mathcal{F}_\rho^cC_{abc}) + \frac{1}{2}\right). \quad (68)$$

The first order action is entirely equivalent to (60) as can be seen by finding the  $\mathcal{F}_\mu^a$  equation of motion and solving it purely algebraically to find  $\mathcal{F}_\mu^a = \partial_\mu X^a$ . Substituting this back into the first order action reproduces (60) exactly.

In the dual description we will exchange the roles of Bianchi identities and equations of motion. The following first order Lagrangian, is duality symmetric,

$$L = \sqrt{-h}\left(\frac{1}{2}h^{\mu\nu}\mathcal{F}_\mu^a\mathcal{F}_\nu^bg_{ab} + \frac{1}{6}\epsilon^{\mu\nu\rho}\mathcal{F}_\mu^a\mathcal{F}_\nu^b\mathcal{F}_\rho^cC_{abc} + \frac{1}{2\sqrt{2}}\epsilon^{\mu\nu\rho}\partial_\mu y_{ab}\mathcal{F}_\nu^a\mathcal{F}_\rho^b - \frac{1}{2}\right). \quad (69)$$

In the above expression  $y_{ab}$ , antisymmetric under  $a \leftrightarrow b$ , is a dual winding co-ordinate. The  $y_{ab}$  equation of motion is now the Bianchi identity (66). Solving it as we did earlier and substituting for  $\mathcal{F}_\mu^a$  leads to the action (60) provided the background admits the appropriate Killing vectors so that  $g_{ab}$  and  $C_{abc}$  are independent of the co-ordinates. Thus provided the Killing vectors exist, the duality symmetric action is equivalent to the original one. The equation of motion for  $\mathcal{F}_\mu^a$  is now

$$h^{\mu\nu}\mathcal{F}_\nu^bg_{ab} + \frac{1}{2}\epsilon^{\mu\nu\rho}\mathcal{F}_\nu^b\mathcal{F}_\rho^cC_{abc} + \frac{1}{\sqrt{2}}\epsilon^{\mu\nu\rho}(\partial_\rho y_{ab})\mathcal{F}_\nu^b = 0. \quad (70)$$

We can re-organize the information about the equations of motion and Bianchi identities by defining

$$\tilde{\mathcal{F}}_{\mu ab} = \partial_{\mu} y_{ab} \quad (71)$$

and writing

$$\begin{pmatrix} \mathcal{G}_{\mu a} \\ \tilde{\mathcal{G}}_{\mu}^{mn} \end{pmatrix} = \begin{pmatrix} g_{ab} + \frac{1}{2} C_a{}^{ef} C_{bef} & \frac{1}{\sqrt{2}} C_a{}^{kl} \\ \frac{1}{\sqrt{2}} C^{mn}{}_b & g^{mn,kl} \end{pmatrix} \begin{pmatrix} \mathcal{F}_{\mu}^b \\ \tilde{\mathcal{F}}_{\mu kl} \end{pmatrix}, \quad (72)$$

where  $g^{mn,kl} = \frac{1}{2}(g^{mk}g^{nl} - g^{ml}g^{nk})$  and has the effect of raising an antisymmetric pair of indices.  $\mathcal{F}_{\mu}^a$  and  $\tilde{\mathcal{F}}_{\mu ab}$  are straight derivatives of the co-ordinates and are therefore rather like displacements, whereas  $\mathcal{G}_{\mu a}$  and  $\tilde{\mathcal{G}}_{\mu}^{ab}$  are rather like field strengths. A generalized displacement can be defined by

$$\mathcal{F}_{\mu}^M = \begin{pmatrix} \mathcal{F}_{\mu}^a \\ \tilde{\mathcal{F}}_{\mu ab} \end{pmatrix} \quad (73)$$

and a generalized field strength by

$$\mathcal{G}_{\mu M} = \begin{pmatrix} \mathcal{G}_{\mu a} \\ \tilde{\mathcal{G}}_{\mu}^{ab} \end{pmatrix} \quad (74)$$

The equation of motion and Bianchi identity can then both be written as

$$\partial_{\mu} \mathcal{G}_M^{\mu} = 0. \quad (75)$$

The field strengths are then related to the displacements by

$$\mathcal{G}_{\mu M} = M_{MN} \mathcal{F}_{\mu}^N. \quad (76)$$

$M_{MN}$  is our generalized metric, explicitly defined by equation (72). This is the generalization of the spacetime metric to include the potential of the 3-form potential to be found in M-theory.

Another, perhaps more intuitive way of thinking of this, is to use the original definitions of  $\mathcal{F}_{\mu}^M$  and  $\mathcal{G}_M^{\mu}$  and re-write the Lagrangian as

$$L = \mathcal{G}_M^{\mu} \mathcal{F}_{\mu}^M. \quad (77)$$

This last form of the action should remind us of the action for magnetostatics in which the role of the magnetic displacement  $B$  is played by  $\mathcal{F}$  and the role of the magnetic field  $H$  by  $\mathcal{G}$ . The generalized metric is then seen to be rather similar to that of the magnetic permeability. The word ‘‘magnetic’’ has been used here because the  $M2$ -brane is presumed to only be moving in spacelike directions and only responds to magnetic components of the field strength. If that were not the case, one could just as well think of electrostatics and make the analogy with  $D$  and  $E$ .

The nature of membrane duality is now apparent. Take the collection of membrane one forms where one has both spacetime and winding co-ordinates:

$$dZ^M = \begin{pmatrix} dx^a \\ dy_{ab} \end{pmatrix} \quad (78)$$

and make a co-ordinate transformation so that

$$dZ \rightarrow \hat{d}Z = T dZ \quad (79)$$

for some constant matrix  $T$ . Then

$$L = dZ^T M dZ \quad (80)$$

gets mapped into

$$L = dZ^T T^T M T dZ = dZ^T \hat{M} dZ \quad (81)$$

where

$$\hat{M} = T^T M T. \quad (82)$$

So as long as  $T$  is invertible  $\hat{M}$  is equivalent to  $M$ . This is what is meant by duality.

To progress further one needs to specify the number of relevant directions since each case is different. The remainder of this paper is devoted to this topic and to understanding how and in what way the duality group has meaning even when no Killing vectors are present.

## 5 Generalized Geometry

In conventional Riemannian geometry, one has a differentiable manifold  $\mathcal{M}$  that has a set of co-ordinates together with a distance function in the form of a metric tensor  $g_{ab}$ . At each point of the manifold, one has the tangent space  $T$  which is spanned by vectors. The metric allows one to calculate the norm of these vectors.

Vector fields  $X$  are then sections of the tangent bundle of the manifold,  $T(\mathcal{M})$ . Suppose that one performs an infinitesimal transformation of the co-ordinates  $x^a \rightarrow x^a + X^a$ . Under such a transformation, a tensor  $S$  will transform to  $S + \delta S$  where  $\delta S = \mathbf{L}_X S$  and  $\mathbf{L}$  is the Lie derivative. The diffeomorphisms when acting on tensors then form an algebra since

$$\mathbf{L}_{X_1} \mathbf{L}_{X_2} - \mathbf{L}_{X_2} \mathbf{L}_{X_1} = \mathbf{L}_{[X_1, X_2]}. \quad (83)$$

In generalized geometry, these structures are enlarged. Courant [36], Hitchin [9] and Gualtieri [10] have shown how to enlarge the space to include  $p$ -forms. In their work, the tangent space at each point is enlarged to include a  $p$ -form; the generalized tangent space being  $T \oplus \Lambda^p(T^*)$ . The generalized metric of the previous section provides the manifold with a generalized metric structure for the specific case of  $p = 2$ . Essentially, the metric has been replaced by an object that contains the original metric together with a 3-form potential  $C$

whose components are  $C_{abc}$  and whose field strength is then  $F = dC$ . From here on, since we are focused on two-branes in M theory, we will only consider the case of  $p = 2$ . Higher dimensional examples occur when there are fivebranes are present and we must have a generalized structure of the form  $T \oplus \Lambda^2(T^*) \oplus \Lambda^5(T^*)$ . An analysis of this and other more complex situations will appear in a future paper. The generalized tangent bundle in the case at hand is  $T(\mathcal{M}) \otimes \Lambda^2 T^*(\mathcal{M})$ . Sections of this bundle,  $A$ , are the sum of vector field  $X$  and a two-form  $\chi$ ,  $A = X \oplus \chi$ . In generalized geometry, one must supplement the co-ordinates  $x^a$  by the winding co-ordinates  $y_{ab}$  introduced in the previous section. Diffeomorphisms are found in the usual way. Reparameterizations of  $x^a$  that are only  $x$ -dependent are the usual co-ordinate transformations. Reparameterizations of  $y_{ab}$  that are only  $x$ -dependent amount to gauge transformations of the  $C$ -field. Suppose one defines a generalized infinitesimal line element  $d\mathfrak{s}$  by

$$d\mathfrak{s}^2 = M_{MN} dZ^M dZ^N \quad (84)$$

where  $Z^M = (x^a, y_{ab})$ . Then if  $y_{ab} \rightarrow y_{ab} + \frac{1}{\sqrt{2}}\lambda_{ab}$  then demanding that the generalized line element remains invariant, induces a gauge transformation  $C_{abc} \rightarrow C_{abc} + \partial_{[a}\lambda_{bc]}$ . These are the usual gauge transformations to be expected of an abelian three form potential. However these gauge transformations are reducible. If  $\lambda$  is exact, then since  $d\lambda = 0$ ,  $C$  is unchanged. Any gauge transformation that is exact has no effect on the fields of the problem. Composition of general transformations shows, following Courant, that these transformations form a Lie algebroid rather than a Lie algebra. This really just a way of saying that there is extra structure beyond the tangent bundle of the manifold; in our case, the space of two-forms. Vectors together with two-forms constitute the gauge transformations of the theory we are interested in. The Courant bracket algebra for the composition of two generalized gauge transformations is

$$[X + \xi, Y + \eta]_C = [X, Y] + \mathbf{L}_X \eta - \mathbf{L}_Y \xi - d(\iota_X \eta - \iota_Y \xi) \quad (85)$$

where the bracket with a suffix  $C$  is the Courant bracket, the bracket with no suffix is the usual Lie bracket, and  $X, Y \in T$  and  $\eta, \xi \in \Lambda^2(T^*)$ . This is almost the same as the algebra diffeomorphisms found in the canonical version of M-theory. The first term on the right in (85) is the usual Lie bracket on expects from commuting two diffeomorphisms. The second and third terms are what one expects from carrying out a diffeomorphism on the two-forms. The last term is an extra contribution to the gauge transformations of the three form. However, since it is exact, it has no effect on the physical fields of the theory. The only discrepancy between (54) and (85), is thus a term involving the field strength.

Suppose now that we take a section of the generalized tangent bundle and twist it by the three-form. Let  $\rho_C$  be the twist operation.

$$(X, \xi) \rightarrow \rho_C(X, \xi) = (X, \xi + \frac{1}{\sqrt{2}}\iota_X C) \quad (86)$$

The idea of the twist operator is to take a background and add some  $C$ -field to it. Thus

$$\rho_{C_1}\rho_{C_2} = \rho_{C_1+C_2} \quad (87)$$

since  $C$  is an abelian field. Then

$$[\rho_C(X, \xi), \rho_C(Y, \eta)]_C = \rho_C([X + \xi, Y + \eta]_C + \frac{1}{\sqrt{2}}\iota_Y\iota_X F - \frac{1}{\sqrt{2}}d\iota_X\iota_Y C). \quad (88)$$

This expression motivates the idea of a Courant bracket twisted by the 3-gerbe  $C$ . Let a bracket with the suffix T be the twisted Courant bracket defined by

$$[X + \xi, Y + \eta]_T = [X + \xi, Y + \eta]_C + \frac{1}{\sqrt{2}}\iota_Y\iota_X F \quad (89)$$

This expression now agrees exactly with the algebra of diffeomorphisms and gauge transformations found in the canonical theory.

One might be concerned that the Jacobi identity fails for such expressions. The Jacobiator of the Courant bracket is defined to be

$$\begin{aligned} J_C(X + \xi, Y + \eta, Z + \chi) &= [[X + \xi, Y + \eta]_C, Z + \chi]_C + [[Y + \eta, Z + \chi]_C, X + \xi]_C \\ &\quad + [[Z + \chi, X + \xi]_C, Y + \eta]_C. \end{aligned} \quad (90)$$

The Jacobiator vanishes if the Jacobi identity holds. Explicit calculation shows that the Jacobiator of the Courant bracket is

$$J_C(X + \xi, Y + \eta, Z + \chi) = -d[\iota_X\iota_Z d\eta + \iota_Y\iota_X d\chi + \iota_Z\iota_Y d\xi] \quad (91)$$

Although the Jacobiator does not vanish, it is an exact two-form and thus has does not lead to a violation of the Jacobi identity when acting on the fields.

Similarly for the twisted Courant bracket. The Jacobiator is then  $J_T$  and is given by

$$J_T = J_C - \frac{1}{\sqrt{2}}\iota_Z\iota_Y\iota_X dF \quad (92)$$

No physically meaningful violations of the Jacobi identity will be encountered as long as the field  $F$  obeys its Bianchi identity,  $dF = 0$ . We therefore conclude that the correct generalization of the Lie bracket is the twisted Courant bracket and this reproduces the algebra of diffeomorphisms and gauge transformations. Perhaps the failure of the Jacobiator to vanish should be taken to mean that there is further structure here to be uncovered. It is tempting to speculate that this will be resolved by understanding precisely what the sectioning condition really is.

Consider now the generalized tangent bundle. At each point in the four-dimensional space, the vectors are in the **4** of  $SL(4)$  and the two-forms are in the **6** of  $SL(4)$ . Suppose that in an

orthonormal frame the vectors have components  $X^a$  and the two-forms have components  $\xi_{ab}$ . Furthermore, suppose that the tangent space is equipped with the norm  $(\delta_{ab}, \frac{1}{2}(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}))$ . Under the action of  $SL(4)$ , the vectors are mapped into vectors and the two-forms into two-forms in the obvious way and their norm will also be preserved. Additionally, one can map vectors into two-forms using a three form, by the operation  $\rho_C$ .  $C$  lies in a **4** of  $SL(4)$ .

There is an inverse operation which maps two-forms into vectors using a tri-vector,  $B^{ijk}$ . Under this map  $(X^i, \xi_{jk}) \rightarrow (X^i + B^{ijk}\xi_{jk}, \xi_{jk})$ .  $B^{ijk}$  is in another **4** of  $SL(4)$ . The central observation is now that these fields that are representations of  $SL(4)$  can be unified into a representation of  $SL(5)$ . By doing so, one has stopped treating the metric and three-form of M theory as different types of object, and put them onto a democratic footing, treating them both as different facets of a single geometric object. The generalized metric acts on a **10** of  $SL(5)$ .  $SL(5)$  can be decomposed into its  $SL(4) \otimes U(1)$  subgroups. Under this decomposition the **10** of  $SL(5)$  is **6**<sub>-2</sub>  $\otimes$  **4**<sub>3</sub> being the vectors and two-forms respectively. The  $U_1$  charges represent the scaling dimension of the vectors and two-forms.

The generalized metric is in the **24**  $\subset$  **10**  $\otimes$  **10** of  $SL(5)$ . Under  $SL(4)$ , the **24** decomposes as **15**<sub>0</sub>  $\oplus$  **4**  $\oplus$  **4**  $\oplus$  **1**. the generalized metric is in fact a parameterization of the symmetric space  $SL(5)/SO(5)$ , much as the reduced metric for general relativity was a parameterization of the symmetric space  $SL_{\bar{d}}/SO(\bar{d})$ . The **15** is the four-dimensional spatial metric, and the **4** is the three-form  $C$ . The **4** corresponds to the trivectors  $B^{ijk}$ , however the generalized metric is independent of these which reflects the local  $SO(5)$  invariance of the system we are considering.

What have developed in this section is a description of the fields of our problem that show how it might be possible to construct a dynamical theory using this unified description motivated by the structure of the U-duality group. In the next sections we carry out this program and discover that the theory has a global  $SL(5)$  symmetry, which can be used to interchange the spatial metric and components of the three form. We also discover that this picture is in no way dependent having compactified any of our spatial dimensions on a torus. It therefore is a tool which transcends our usual notions of U-duality.

What we have not done, and this will be the subject of a future publication, is to exhibit the full power of the possibility of mixing  $x^i$  and  $y_{ij}$  coordinates. Also, we fully expect that similar considerations will work in higher dimensions, although we are already aware of a number of complications to a straightforward extension our treatment. This too will be the subject of a future publication.

## 6 Canonical M-Theory Revisited

Now that we have the appropriate constituent elements, we can attempt to reconstruct the dynamical theory out of the generalized metric. Following the canonical Hamiltonian formalism, there are two types of the term. The kinetic terms which are essentially the momentum squared and the potential terms given only in terms of the fields and their spatial derivatives.

Let us begin with the potential terms. The potential in terms of the generalized metric will be given by:

$$V = \gamma^{1/2} \left( \frac{1}{12} M^{MN} (\partial_M M^{KL}) (\partial_N M_{KL}) - \frac{1}{2} M^{MN} (\partial_N M^{KL}) (\partial_L M_{MK}) + \frac{1}{12} M^{MN} (M^{KL} \partial_M M_{KL}) (M^{RS} \partial_N M_{RS}) + \frac{1}{4} M^{MN} M^{PQ} (M^{RS} \partial_P M_{RS}) (\partial_M M_{NQ}) \right) \quad (93)$$

where  $\partial_M = (\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y_{ab}})$ .  $V$  is then evaluated using the definition of the generalized metric  $M_{MN}$  described in section four by equations (72) and (76), in terms of the spatial metric and three form with the assumption that  $\partial_y = 0$ . We use just the spatial part of the metric in the computation of  $V$  since we are only allowing spacelike duality transformations. in the synchronous gauge, this means replacing  $g_{ab}$  by  $\gamma_{ij}$ . After a long and careful calculation, the result, up to a total derivative, is

$$V = \gamma^{1/2} (R(\gamma) - \frac{1}{48} F^2). \quad (94)$$

This is exactly the potential term we had for eleven dimensional supergravity. Note, that at first sight  $V$  appears to stand little chance of being related to the Ricci scalar since it does not contain any terms second order in derivatives (as compared with terms quadratic in first derivatives). However the key is to take the usual Einstein Hilbert term and integrate by parts any terms second order in derivatives. This produces a Lagrangian of the type given in  $V$ . This is also why the equivalence is only up to surface terms. It should also be noted that  $V$  is thus a  $SL(5)$  scalar times  $\gamma^{1/2}$ . Strictly speaking this means that only  $SL(5)$  transformations that preserve  $\gamma$  are invariances of  $V$ . It is however probably the case, that once a proper section condition is found, one will be able to extend the invariance to include those where  $\gamma$  is varying. That is because the factor of  $\gamma^{1/2}$  more properly belongs with the  $d^{\bar{d}}x$  to create an invariant volume measure. We currently lack a formulation that includes an invariant volume measure on both the  $x$  and  $y$  co-ordinates.

It is also worth making the comment here that the first guess for such a term would be the Ricci scalar of the generalized metric. This does not give the right answer. Apart from numerical factors not working out, crucially, one does not obtain the gauge invariant field strength  $F$  from  $C$ . This is a consequence of the traditional Ricci scalar being constructed to be a scalar under diffeomorphisms. The reason is as we have seen in the section three where we described the constraint algebra, when there is a nontrivial  $C$  field, the constraints generate more than just the diffeomorphisms. This also probably related to the fact that the beta function of the doubled string theory [22] does not give the usual Ricci curvature of the doubled space.

Thus one suspects we need a generalized notion of curvature that respects all of the symmetries. The recent work of Hohm, Hull and Zwiebach [38] reaches similar conclusions for the doubled geometry of the string. There they use insights from string field theory. Here we simply have guessed the answer and checked it reduces to the usual result once the dependence of the  $y$  co-ordinates is removed. Removing the  $y$  dependence is essentially our choice of how to take a section in the generalized space. The rules for how to pick a section more generally are something that needs to be explored. Nevertheless it is clear that this choice should produce the usual description of supergravity, and it does.

It is also important to note that it is equivalent only up to a total derivative term. This is because to show the equivalence we had to integrate the usual Einstein-Hilbert term by parts to remove the terms that were second order in derivatives. Our potential term only contains first derivatives on the fields so this trick of removing total derivatives was necessary to stand any chance of there being an equivalence between the two.

This suggests that there may be terms second order in derivatives that can complete the above potential to really give something like a generalized scalar curvature.

We are thus now left with the kinetic pieces. In terms of the *generalized metric*,  $M_{IJ}$  we find the following simple kinetic term,

$$T = -\gamma^{1/2} \frac{1}{12} ((\text{tr } M^{-1} \dot{M}) + (\text{tr } M^{-1} \dot{M})^2) \quad (95)$$

and we discover when this is evaluated in terms of the original fields that this is identical to the kinetic term for  $g_{ij}$  and  $C_{ijk}$  in the usual canonical formalism given in section three. Of course to see the equivalence one has to take the kinetic term for M theory and replace the momenta by the time derivatives of the fields. In (95) the second term is the piece that is required to reproduce the “timelike” direction for the trace of the metric. If the  $V = 0$  we see that considerations that parallel section two show that the metric is just geodesic motion on the coset space  $SL(5)/SO(5)$ .

Thus we can reproduce the entire Hamiltonian in a way has a manifest global  $SL(5)$  invariance simply as  $T + V$ , apart from the remote possibility of a problem with factors of  $\gamma$ .

Lastly, the fact that the generalized geometry construction of  $V$  reproduces  $R - \frac{1}{48}F^2$  indicates that it might be possible to include timelike directions within this formulation. Under such circumstances, if the four dimensions are Lorentzian, then the coset space becomes  $SL(5)/SO(3, 2)$ , [39]. However, since our treatment includes the conventional duality picture, the possibility of having complex fluxes and also strange signatures with multiple timelike directions for spacetime, [16] discourages us from pursuing such a possibility energetically at present.

## 7 Conclusions and Speculations

First let us restate the main result. The Hamiltonian of gravity and a three form potential has been reformulated using generalized geometry in a manifestly  $SL_5$  invariant way with no dimensional reduction of the theory.

We expect a similar treatment to be possible to make manifest the higher dimensional duality groups of M-theory. Of course it will be very interesting to see using this approach what happens for  $E_8$  and beyond.

It has been a very pleasant exercise to see the utility and naturalness of generalized geometry in this approach, without this geometric perspective such a construction would not have been possible. One can also imagine that the dynamics described here may be useful in providing some notion of curvature in generalized geometry.

In this regard one aspect immediately come to the fore. First, the equivalence was only shown up to boundary terms. In formulating a proper curvature these neglected terms may well be important. Physically these terms would also be interesting in discussing things like black hole thermodynamics where the usual boundary terms (ie Gibbons Hawking terms) provide a useful way of evaluating the free energy of a black hole. Whether the boundary terms can also be written in a manifestly duality invariant way using the generalized metric will be crucial to understanding the connection between black holes and duality groups in supergravity.

The boundary terms also allude to bigger issues regarding topological restrictions in this construction. Essentially we have ignored them and so the duality group should be viewed as a solution generating symmetry group for supergravity rather than as an exact equivalence of the theory in those backgrounds. A detailed understanding of winding modes and quantization of charges should restrict the duality group to being integer valued and then there should be equivalence just as for T-duality.

Another open question is to determine the section condition. Our choice here is certainly not the only one and in particular the topological restrictions on how to implement this choice will be crucial for the construction of U-folds. Related to this is how one should construct the measure over the full generalized space. That is we have worked with densities throughout that need to be supplemented by  $\sqrt{\det(\gamma)}$  factors in integrals. Since through our section condition we have removed the need to integrate over the  $y$  co-ordinates this is not an issue but a full duality invariant treatment would require a duality invariant measure over the full space.

As indicated in the body of the text, incorporating time in this scheme will be difficult since the dualities involving timelike directions are more exotic and yet it seems that this should be possible.

Other ways in which this approach might prove useful is to try and construct the higher order correction terms in curvature. Essentially the idea would be to use the duality group to restrict these terms. This has been useful in IIB string theory using the S-duality and it seems that a similar approach may be useful in M-theory. Finally of course, it will be interesting to see how fermions fit into this picture. All of these topics we intend to return to in future

works.

## 8 Acknowledgements

DSB wishes to thank DAMTP, University of Cambridge for continued hospitality and is supported in part by the Queen Mary STFC rolling grant ST/G000565/1. We would like to thank the Mitchell family for their generous hospitality at Cook's Branch Nature conservancy where some of this work was carried out. MJP would like to thank Mitchell foundation and Trinity College Cambridge for their support. We are grateful to the following people who made comments on aspects of the manuscript after the initial preprint appeared: Hadi Godazgar, Chris Hull, Bernard Julia, Emanuel Malek and Hermann Nicolai.

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